

# RATIONALITY AND BRAUER GROUP OF A MODULI SPACE OF FRAMED BUNDLES

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**ABSTRACT.** We prove that the moduli spaces of framed bundles over a smooth projective curve are rational. We compute the Brauer group of these moduli spaces to be zero under some assumption on the stability parameter.

## 1. INTRODUCTION

Let  $X$  be a compact connected Riemann surface of genus  $g$ , with  $g \geq 2$ . A framed bundle on  $X$  is a pair of the form  $(E, \phi)$ , where  $E$  is a vector bundle on  $X$ , and

$$\phi : E_{x_0} \longrightarrow \mathbb{C}^r$$

is a non-zero  $\mathbb{C}$ -linear homomorphism, where  $r$  is the rank of  $E$ . The notion of a (semi)stable vector bundle extends to that for a framed bundle. But the (semi)stability condition depends on a parameter  $\tau \in \mathbb{R}_{>0}$ . Fix a positive integer  $r$ , and also fix a holomorphic line bundle  $L$  over  $X$ . Also, fix a positive number  $\tau \in \mathbb{R}$ . Let  $\mathcal{M}_L^\tau(r)$  be the moduli space of  $\tau$ -stable framed bundles of rank  $r$  and determinant  $L$ .

In [BGM], we investigated the geometric structure of the variety  $\mathcal{M}_L^\tau(r)$ . The following theorem was proved in [BGM]:

*Assume that  $\tau \in (0, \frac{1}{(r-1)!(r-1)})$ . Then the isomorphism class of the Riemann surface  $X$  is uniquely determined by the isomorphism class of the variety  $\mathcal{M}_L^\tau(r)$ .*

Our aim here is to investigate the rationality properties of the variety  $\mathcal{M}_L^\tau(r)$ . We prove the following (see Theorem 2.3 and Corollary 3.2):

*The variety  $\mathcal{M}_L^\tau(r)$  is rational.*

*If  $\tau \in (0, \frac{1}{(r-1)!(r-1)})$ , then*

$$\mathrm{Br}(\mathcal{M}_L^\tau(r)) = 0,$$

*where  $\mathrm{Br}(\mathcal{M}_L^\tau(r))$  is the Brauer group of  $\mathcal{M}_L^\tau(r)$ .*

The rationality of  $\mathcal{M}_L^\tau(r)$  is proved by showing that  $\mathcal{M}_L^\tau(r)$  is birational to the total space of a vector bundle over the moduli space of stable vector bundles  $E$  on  $X$  together with a line in the fiber of  $E$  over a fixed point. The rationality of  $\mathcal{M}_L^\tau(r)$  also follows from [Ho1], Example 6.9, taking  $D$  to be the point  $x_0$ .

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The Brauer group of  $\mathcal{M}_L^\tau(r)$  is computed by considering the morphism to the usual moduli space that forgets the framing.

## 2. RATIONALITY OF MODULI SPACE

Let  $X$  be a compact connected Riemann surface of genus  $g$ , with  $g \geq 2$ . Fix a holomorphic line bundle  $L$  over  $X$ , and take an integer  $r > 0$ . Fix a point  $x_0 \in X$ . A framed coherent sheaf over  $X$  is a pair of the form  $(E, \phi)$ , where  $E$  is a coherent sheaf on  $X$  of rank  $r$ , and

$$\phi : E_{x_0} \longrightarrow \mathbb{C}^r$$

is a non-zero  $\mathbb{C}$ -linear homomorphism. Let  $\tau > 0$  be a real number. A framed coherent sheaf is called  $\tau$ -stable (respectively,  $\tau$ -semistable) if for all proper subsheaves  $E' \subset E$ , we have

$$(2.1) \quad \deg E' - \epsilon(E', \phi)\tau < \operatorname{rk} E' \frac{\deg E - \tau}{\operatorname{rk} E}$$

(respectively,  $\deg E' - \epsilon(E', \phi)\tau \leq \operatorname{rk} E'(\deg E - \tau)/\operatorname{rk} E$ ), where

$$\epsilon(E', \phi) = \begin{cases} 1 & \text{if } \phi|_{E'_{x_0}} \neq 0, \\ 0 & \text{if } \phi|_{E'_{x_0}} = 0. \end{cases}$$

A framed bundle is a framed coherent sheaf  $(E, \phi)$  such that  $E$  is locally free.

We remark that the framed coherent sheaves considered here are special cases of the objects considered in [HL], and hence from [HL] we conclude that the moduli space  $\mathcal{M}_L^\tau(r)$  of  $\tau$ -stable framed bundles of rank  $r$  and determinant  $L$  is a smooth quasi-projective variety.

Let  $(E, \phi)$  be a  $\tau$ -semistable framed coherent sheaf. We note that if  $\tau < 1$ , then  $E$  is necessarily torsion-free, because a torsion subsheaf of  $E$  will contradict  $\tau$ -semistability, hence in this case  $E$  is locally free. But if  $\tau$  is large, then  $E$  can have torsion. In particular, the natural compactification of  $\mathcal{M}_L^\tau(r)$  using  $\tau$ -semistable framed coherent sheaves could have points which are not framed bundles.

**Lemma 2.1.** *There is a dense Zariski open subset*

$$(2.2) \quad \mathcal{M}_L^\tau(r)^0 \subset \mathcal{M}_L^\tau(r)$$

*corresponding to pairs  $(E, \phi)$  such that  $E$  is a stable vector bundle of rank  $r$ , and  $\phi$  is an isomorphism.*

*The moduli space  $\mathcal{M}_L^\tau(r)$  is irreducible.*

*Proof.* From the openness of the stability condition it follows immediately that the locus of framed bundles  $(E, \phi)$  such that  $E$  is not stable is a closed subset of the moduli space  $\mathcal{M}_L^\tau(r)$  (see [Ma, p. 635, Theorem 2.8(B)] for the openness of the stability condition). It is easy to check that the locus of framed bundles  $(E, \phi)$  such that  $\phi$  is not an isomorphism is a closed subset of  $\mathcal{M}_L^\tau(r)$ . Therefore,  $\mathcal{M}_L^\tau(r)^0$  is a Zariski open subset of  $\mathcal{M}_L^\tau(r)$ .

We will now show that this open subset  $\mathcal{M}_L^\tau(r)^0$  is dense. Let  $(E, \varphi)$  be a  $\tau$ -stable framed bundle. The moduli stack of stable vector bundles is dense in the moduli stack of coherent sheaves, and both stacks are irreducible (see, for instance, [Ho2, Appendix]). Therefore we can construct a family  $\{E_t\}_{t \in T}$  of vector bundles parametrized by an irreducible smooth curve  $T$  with a base point  $0 \in T$  such that the following two conditions hold:

- (1)  $E_0 \cong E$ , and
- (2) the vector bundle  $E_t$  is stable for all  $t \neq 0$ .

Shrinking  $T$  if necessary (by taking a nonempty Zariski open subset of  $T$ ), we get a family of frames  $\{\phi_t\}_{t \in T}$  such that  $\phi_0$  is the given frame  $\phi$ , and  $\phi_t : E_{t, x_0} \rightarrow \mathbb{C}^r$  is an isomorphism for all  $t \neq 0$ . Since  $E_t$  is stable, and  $\phi_t$  is an isomorphism, it is easy to check that  $(E_t, \phi_t)$  is  $\tau$ -stable. Therefore,  $\mathcal{M}_L^\tau(r)^0$  is dense in  $\mathcal{M}_L^\tau(r)$ .

To prove that  $\mathcal{M}_L^\tau(r)$  is irreducible, first note that  $\mathcal{M}_L^\tau(r)^0$  is irreducible because the moduli stack of stable vector bundles of fixed rank and determinant is irreducible. Since  $\mathcal{M}_L^\tau(r)^0 \subset \mathcal{M}_L^\tau(r)$  is dense, it follows that  $\mathcal{M}_L^\tau(r)$  is irreducible.  $\square$

Let  $\mathcal{N}_P$  be the moduli space of pairs of the form  $(E, \ell)$ , where  $E$  is a stable vector bundle on  $X$  of rank  $r$  with determinant  $L$ , and  $\ell \subset E_{x_0}$  is a line. Consider  $\mathcal{M}_L^\tau(r)^0$  defined in (2.2). Let

$$(2.3) \quad \beta : \mathcal{M}_L^\tau(r)^0 \longrightarrow \mathcal{N}_P$$

be the morphism defined by  $(E, \phi) \mapsto (E, \phi^{-1}(\mathbb{C} \cdot e_1))$ , where the standard basis of  $\mathbb{C}^r$  is denoted by  $\{e_1, \dots, e_r\}$ .

**Proposition 2.2.** *The variety  $\mathcal{M}_L^\tau(r)^0$  is birational to the total space of a vector bundle over  $\mathcal{N}_P$ .*

*Proof.* We will first construct a tautological vector bundle over  $\mathcal{N}_P$ . Let  $\mathcal{N}_L(r)$  be the moduli space of stable vector bundles on  $X$  of rank  $r$  and determinant  $L$ . Consider the projection

$$(2.4) \quad f : \mathcal{N}_P \longrightarrow \mathcal{N}_L(r)$$

defined by  $(E, \ell) \mapsto E$ . Let  $P_{\text{PGL}} \rightarrow \mathcal{N}_L(r)$  be the principal  $\text{PGL}(r, \mathbb{C})$ -bundle corresponding to  $f$ ; the fiber of  $P_{\text{PGL}}$  over any  $E \in \mathcal{N}_L(r)$  is the space of all linear isomorphisms from  $P(\mathbb{C}^r)$  (the space of lines in  $\mathbb{C}^r$ ) to  $P(E_{x_0})$  (the space of lines in  $E_{x_0}$ ); since the automorphism group of  $E$  is the nonzero scalar multiplications (recall that  $E$  is stable), the projective space  $P(E_{x_0})$  is canonically defined by the point  $E$  of  $\mathcal{N}_L(r)$ . Let

$$Q \subset \text{PGL}(r, \mathbb{C})$$

be the maximal parabolic subgroup that fixes the point of  $P(\mathbb{C}^r)$  representing the line  $\mathbb{C} \cdot e_1$ . The principal  $\text{PGL}(r, \mathbb{C})$ -bundle

$$f^* P_{\text{PGL}} \longrightarrow \mathcal{N}_P$$

has a tautological reduction of structure group

$$\tilde{E}_Q \subset f^* P_{\text{PGL}}$$

to the parabolic subgroup  $Q$ ; the fiber of  $\tilde{E}_Q$  over any point  $(E, \ell) \in \mathcal{N}_P$  is the space of all linear isomorphisms

$$\rho : P(\mathbb{C}^r) \longrightarrow P(E_{x_0})$$

such that  $\rho(\mathbb{C} \cdot e_1) = \ell$ . The standard action of  $\text{GL}(r, \mathbb{C})$  on  $\mathbb{C}^r$  defines an action of  $Q$  on  $(\mathbb{C} \cdot e_1)^* \otimes_{\mathbb{C}} \mathbb{C}^r$ . Let

$$(2.5) \quad W := f^* P_{\text{PGL}}((\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^r) \longrightarrow \mathcal{N}_P$$

be the vector bundle over  $\mathcal{N}_P$  associated to the principal  $\text{PGL}(r, \mathbb{C})$ -bundle  $f^* P_{\text{PGL}}$  for the above  $\text{PGL}(r, \mathbb{C})$ -module  $(\mathbb{C} \cdot e_1)^* \otimes_{\mathbb{C}} \mathbb{C}^r$ . The action of  $Q$  on  $(\mathbb{C} \cdot e_1)^* \otimes_{\mathbb{C}} \mathbb{C}^r$  fixes

$$\text{Id}_{\mathbb{C} \cdot e_1} \in (\mathbb{C} \cdot e_1)^* \otimes_{\mathbb{C}} \mathbb{C}^r = \text{Hom}(\mathbb{C} \cdot e_1, \mathbb{C}^r).$$

Therefore, the element  $\text{Id}_{\mathbb{C} \cdot e_1}$  defines a nonzero section

$$(2.6) \quad \sigma \in H^0(\mathcal{N}_P, W),$$

where  $W$  is the vector bundle in (2.5). Note that the fiber of  $W$  over  $(E, \ell)$  is  $\ell^* \otimes E_{x_0}$ , and the evaluation of  $\sigma$  at  $(E, \ell)$  is  $\text{Id}_{\ell}$ .

The projective bundle  $P(W) \longrightarrow \mathcal{N}_P$  parametrizing lines in  $W$  is identified with the pullback  $f^* \mathcal{N}_P$  of the projective bundle  $\mathcal{N}_P$  to the total space of  $\mathcal{N}_P$ , where  $f$  is constructed in (2.4). The tautological section  $\mathcal{N}_P \longrightarrow f^* \mathcal{N}_P$  of the projection  $f^* \mathcal{N}_P \longrightarrow \mathcal{N}_P$  coincides with the section given by  $\sigma$  in (2.6).

Let  $U \subset \mathcal{N}_P$  be some nonempty Zariski open subset such that there exists

$$V \subset W|_U,$$

a direct summand of the line subbundle of  $W|_U$  generated by  $\sigma$ . Consider the vector bundle

$$\mathcal{W} := V^* \otimes_{\mathbb{C}} \mathbb{C}^r \longrightarrow U.$$

The total space of  $\mathcal{W}$  will also be denoted by  $\mathcal{W}$ . Consider the map  $\beta$  defined in (2.3). Let

$$\gamma : \mathcal{M}_L^r(r)^0 \supset \beta^{-1}(U) \longrightarrow \mathcal{W}$$

be the morphism that sends any  $y := (E, \phi) \in \beta^{-1}(U)$  to the homomorphism

$$V_{\beta(y)} \longrightarrow \mathbb{C}^r$$

defined by  $v \longmapsto \phi(v)/\lambda$ , where  $\lambda \in \mathbb{C}^* - \{0\}$  satisfies the identity  $\phi(\sigma(\beta(y))) = \lambda \cdot e_1$ . The morphism  $\gamma$  is clearly birational.  $\square$

**Theorem 2.3.** *The moduli space  $\mathcal{M}_L^r(r)$  is rational.*

*Proof.* Since any vector bundle is Zariski locally trivial, the total space of a vector bundle of rank  $n$  over  $\mathcal{N}_P$  is birational to  $\mathcal{N}_P \times \mathbb{A}^n$ . Therefore, from Proposition 2.2 we conclude that  $\mathcal{M}_L^\tau(r)^0$  is birational to  $\mathcal{N}_P \times \mathbb{A}^n$ , where  $n = \dim \mathcal{M}_L^\tau(r)^0 - \dim \mathcal{N}_P$ .

The variety  $\mathcal{N}_P$  is known to be rational [BY, p. 472, Theorem 6.2]. Hence  $\mathcal{N}_P \times \mathbb{A}^n$  is rational, implying that  $\mathcal{M}_L^\tau(r)^0$  is rational. Now from Lemma 2.1 we infer that  $\mathcal{M}_L^\tau(r)$  is rational.  $\square$

### 3. BRAUER GROUP OF MODULI OF FRAMED BUNDLES

We quickly recall the definition of Brauer group of a variety  $Z$ . Using the natural isomorphism  $\mathbb{C}^r \otimes \mathbb{C}^{r'} \xrightarrow{\sim} \mathbb{C}^{rr'}$ , we have a homomorphism  $\mathrm{PGL}(r, \mathbb{C}) \times \mathrm{PGL}(r', \mathbb{C}) \rightarrow \mathrm{PGL}(rr', \mathbb{C})$ . So a principal  $\mathrm{PGL}(r, \mathbb{C})$ -bundle  $\mathbb{P}$  and a principal  $\mathrm{PGL}(r', \mathbb{C})$ -bundle  $\mathbb{P}'$  on  $Z$  together produce a principal  $\mathrm{PGL}(rr', \mathbb{C})$ -bundle on  $Z$ , which we will denote by  $\mathbb{P} \otimes \mathbb{P}'$ . The two principal bundles  $\mathbb{P}$  and  $\mathbb{P}'$  are called *equivalent* if there are vector bundles  $V$  and  $V'$  on  $Z$  such that the principal bundle  $\mathbb{P} \otimes \mathbb{P}(V)$  is isomorphic to  $\mathbb{P}' \otimes \mathbb{P}(V')$ . The equivalence classes form a group which is called the *Brauer group* of  $Z$ . The addition operation is defined by the tensor product, and the inverse is defined to be the dual projective bundle. The Brauer group of  $Z$  will be denoted by  $\mathrm{Br}(Z)$ .

As before, fix  $r$  and  $L$ . Define

$$\tau(r) := \frac{1}{(r-1)!(r-1)}.$$

Henceforth, we assume that

$$\tau \in (0, \tau(r)),$$

where  $\tau$  is the parameter in the definition of a (semi)stable framed bundle. As before, let  $\mathcal{M}_L^\tau(r)$  be the moduli space of  $\tau$ -stable framed bundles of rank  $r$  and determinant  $L$ .

Let  $\overline{\mathcal{N}}_L(r)$  be the moduli space of semistable vector bundles on  $X$  of rank  $r$  and determinant  $L$ . As in the previous section, the moduli space of stable vector bundles on  $X$  of rank  $r$  and determinant  $L$  will be denoted by  $\mathcal{N}_L(r)$ .

If  $E$  is a stable vector bundle of rank  $r$  and determinant  $L$ , then for any nonzero homomorphism

$$\phi : E_{x_0} \rightarrow \mathbb{C}^r,$$

the framed bundle  $(E, \phi)$  is  $\tau$ -stable (see [BGM, Lemma 1.2(ii)]). Also, if  $(E, \phi)$  is any  $\tau$ -stable framed bundle, then  $E$  is semistable [BGM, Lemma 1.2(i)]. Therefore, we have a morphism

$$(3.1) \quad \delta : \mathcal{M}_L^\tau(r) \rightarrow \overline{\mathcal{N}}_L(r)$$

defined by  $(E, \phi) \rightarrow E$ . Define

$$(3.2) \quad \mathcal{M}_L^\tau(r)' := \delta^{-1}(\mathcal{N}_L(r)) \subset \mathcal{M}_L^\tau(r),$$

where  $\delta$  is the morphism in (3.1). From the openness of the stability condition (mentioned in the proof of Lemma 2.1) it follows that  $\mathcal{M}_L^\tau(r)'$  is a Zariski open subset of  $\mathcal{M}_L^\tau(r)$ .

**Lemma 3.1.** *The Brauer group of the variety  $\mathcal{M}_L^\tau(r)'$  vanishes.*

*Proof.* We noted above that  $(E, \phi)$  is  $\tau$ -stable if  $E$  is stable. Therefore, the morphism

$$\delta_1 := \delta|_{\mathcal{M}_L^\tau(r)'} : \mathcal{M}_L^\tau(r)' \longrightarrow \mathcal{N}_L(r)$$

defines a projective bundle over  $\mathcal{N}_L(r)$ , where  $\delta$  is constructed in (3.1); for notational convenience, this projective bundle  $\mathcal{M}_L^\tau(r)'$  will be denoted by  $\mathcal{P}$ . The homomorphism

$$\delta_1^* : \text{Br}(\mathcal{N}_L(r)) \longrightarrow \text{Br}(\mathcal{P})$$

is surjective, and the kernel of  $\delta_1^*$  is generated by the Brauer class

$$\text{cl}(\mathcal{P}) \in \text{Br}(\mathcal{N}_L(r))$$

of the projective bundle  $\mathcal{P}$  (see [Ga, p. 193]). In other words, we have an exact sequence

$$(3.3) \quad \mathbb{Z} \cdot \text{cl}(\mathcal{P}) \longrightarrow \text{Br}(\mathcal{N}_L(r)) \xrightarrow{\delta_1^*} \text{Br}(\mathcal{M}_L^\tau(r)') \longrightarrow 0.$$

Let

$$\mathbb{P} := \mathcal{N}_L(r) \times P(\mathbb{C}^r) \longrightarrow \mathcal{N}_L(r)$$

be the trivial projective bundle over  $\mathcal{N}_L(r)$ . Consider the projective bundle

$$f : \mathcal{N}_P \longrightarrow \mathcal{N}_L(r)$$

in (2.4). Let

$$(\mathcal{N}_P)^* \longrightarrow \mathcal{N}_L(r)$$

be the dual projective bundle; so the fiber of  $(\mathcal{N}_P)^*$  over any point  $z \in \mathcal{N}_L(r)$  is the space of all hyperplanes in the fiber of  $\mathcal{N}_P$  over  $z$ . It is easy to see that

$$(3.4) \quad \mathcal{P} = (\mathcal{N}_P)^* \otimes \mathbb{P}$$

(the tensor product of two projective bundles was defined at the beginning of this section).

Since  $\mathbb{P}$  is a trivial projective bundle, from (3.4) it follows that

$$\text{cl}(\mathcal{P}) = \text{cl}((\mathcal{N}_P)^*) = -\text{cl}(\mathcal{N}_P) \in \text{Br}(\mathcal{N}_L(r)).$$

But the Brauer group  $\text{Br}(\mathcal{N}_L(r))$  is generated by  $\text{cl}(\mathcal{N}_P)$  [BBGN, Proposition 1.2(a)]. Hence  $\text{cl}(\mathcal{P})$  generates  $\text{Br}(\mathcal{N}_L(r))$ . Now from (3.3) we conclude that  $\text{Br}(\mathcal{M}_L^\tau(r)') = 0$ .  $\square$

**Corollary 3.2.** *The Brauer group of the moduli space  $\mathcal{M}_L^\tau(r)$  vanishes.*

*Proof.* Since  $\mathcal{M}_L^\tau(r)'$  is a nonempty Zariski open subset of  $\mathcal{M}_L^\tau(r)$ , the homomorphism

$$\text{Br}(\mathcal{M}_L^\tau(r)) \longrightarrow \text{Br}(\mathcal{M}_L^\tau(r)')$$

induced by the inclusion  $\mathcal{M}_L^\tau(r)' \hookrightarrow \mathcal{M}_L^\tau(r)$  is injective. Therefore, from Lemma 3.1 it follows that  $\text{Br}(\mathcal{M}_L^\tau(r)) = 0$ .  $\square$

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